Functional correctness via refinement

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Outline

1. Abstract Data Types
2. Refinement
3. ADT Transition Systems
4. Phrasing refinement conditions in VCC

ADT type

An *ADT type* is a finite set N of *operation names*.

- Each operation name n in N has an associated *input type* I_n and an *output type* O_n, each of which is simply a set of values.
- We require that there is a special *exceptional value* denoted by e, which belongs to each output type O_n; and that the set of operations N includes a designated *initialization operation* called *init*.

ADT definition

A (deterministic) *ADT* of type N is a structure of the form

$$\mathcal{A} = (Q, U, E, \{op_n\}_{n \in N})$$

where

- Q is the set of states of the ADT,
- $U \in Q$ is an arbitrary state in Q used as an *uninitialized* state,
- $E \in Q$ is an *exceptional* state.
- Each op_n is a *realisation* of the operation n given by $op_n : Q \times I_n \to Q \times O_n$ such that $op_n(E, -) = (E, e)$ and $op_n(p, a) = (q, e) \implies q = E$.
- Further, we require that the *init* operation depends only on its argument and not on the originating state: thus $init(p, a) = init(q, a)$ for each $p, q \in Q \setminus \{E\}$ and $a \in I_{init}$.

ADT type example: Queue

QType

ADT type $QType = \{init, enq, deq\}$ with

$$
\begin{array}{rcl}
I_{init} & = & \{nil\}, \\
O_{init} & = & \{ok, e\}, \\
I_{enq} & = & \mathbb{B}, \\
O_{enq} & = & \{ok, fail, e\}, \\
I_{deq} & = & \{nil\}, \\
O_{deq} & = & \mathbb{B} \cup \{fail, e\}.
\end{array}
$$

Here $\mathbb{B}$ is the set of bit values $\{0, 1\}$, and *nil* is a "dummy" argument for the operations *init* and *deq*.

ADT example: Queue (parameterized by length k) of type $QType$

$QADT_k$

$$QADT_k = (Q, U, E, \{op_n\}_{n \in QType}) \text{ where}$$

$$Q = \{\epsilon\} \cup \bigcup_{i=1}^{k} \mathbb{B}^i \cup \{E\}$$

$$op_{init}(q, a) = \begin{cases} (\epsilon, ok) & \text{if } q \neq E \\ (E, e) & \text{otherwise.} \end{cases}$$

$$op_{enq}(q, a) = \begin{cases} (q \cdot a, ok) & \text{if } q \neq E \text{ and } |q| < k \\ (E, e) & \text{otherwise.} \end{cases}$$

$$op_{deq}(q, a) = \begin{cases} (q', b) & \text{if } q \neq E \text{ and } q = b \cdot q' \\ (E, e) & \text{otherwise.} \end{cases}$$

Language of sequences of operation calls of an ADT

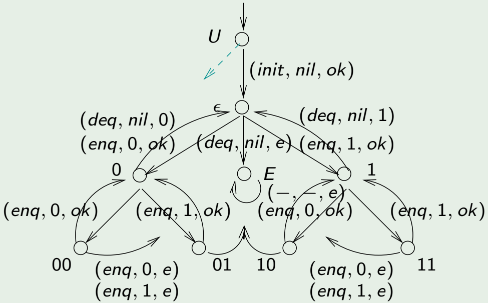
- An ADT $\mathcal{A} = (Q, U, E, \{op_n\}_{n \in N})$ of type N induces a (deterministic) transition system $\mathcal{S}_\mathcal{A} = (Q, \Sigma_N, U, \Delta)$ where
  - $\Sigma_N = \{(n, a, b) \mid n \in N, a \in I_n, b \in O_n\}$ is the set of *operation call* labels corresponding to the ADT type N. The action label (n, a, b) represents a call to operation n with input a that returns the value b.
  - Δ is given by

$$(p, (n, a, b), q) \in \Delta \text{ iff } op_n(p, a) = (q, b).$$

- We define the language of *initialised sequences of operation calls* of $\mathcal{A}$, denoted $L_{init}(\mathcal{A})$, to be $L(\mathcal{S}_\mathcal{A}) \cap ((init, -, -) \cdot \Sigma_N^*)$.
- We say a sequence of operation calls w is *exception-free* if no call in it returns the exceptional value e (i.e. w does not contain a call of the form $(-, -, e)$).

Example: Transition system induced by $QADT_2$

TS induced by $QADT_2$



Refinement between ADT's

Let $\mathcal{A}$ and $\mathcal{B}$ be ADT's of type N. We say $\mathcal{B}$ refines $\mathcal{A}$, written

$$\mathcal{B} \preceq \mathcal{A},$$

iff each exception-free sequence in $L_{init}(\mathcal{A})$ is also in $L_{init}(\mathcal{B})$.

Examples of refinement:

- $QADT_3$ refines $QADT_2$.
- Let $QADT_2'$ be the version of $QADT_2$ where we check for emptiness/fullness of queue and return *fail* instead of e. Then $QADT_2'$ refines $QADT_2$.

Transitivity of refinement

It follows immediately from its definition that refinement is transitive:

Proposition

Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be ADT's of type N, such that $\mathcal{C} \preceq \mathcal{B}$, and $\mathcal{B} \preceq \mathcal{A}$. Then $\mathcal{C} \preceq \mathcal{A}$.

Refinement Condition (RC)

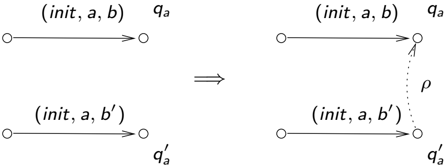
Let $\mathcal{A} = (Q, U, E, \{op_n\}_{n \in N})$ and $\mathcal{A}' = (Q', U', E', \{op_n\}_{n \in N})$ be ADT's of type N. We formulate an *equivalent* condition for $\mathcal{A}'$ to refine $\mathcal{A}$, based on an "abstraction relation" that relates states of $\mathcal{A}'$ to states of $\mathcal{A}$. We say $\mathcal{A}$ and $\mathcal{A}'$ satisfy condition (RC) if there exists a relation $\rho \subseteq Q' \times Q$ such that:

(init) Let $a \in I_{init}$ and let (q_a, b) and (q'_a, b') be the resultant states and outputs after an $init(a)$ operation in $\mathcal{A}$ and $\mathcal{A}'$ respectively, with $b \neq e$. Then we require that $b = b'$ and $(q'_a, q_a) \in \rho$.

(sim) For each $n \in N$, $a \in I_n$, $b \in O_n$, and $p' \in Q'$, with $(p', p) \in \rho$, whenever $p \xrightarrow{(n,a,b)} q$ with $b \neq e$, then there exists $q' \in Q'$ such that $p' \xrightarrow{(n,a,b)} q'$ with $(q', q) \in \rho$.

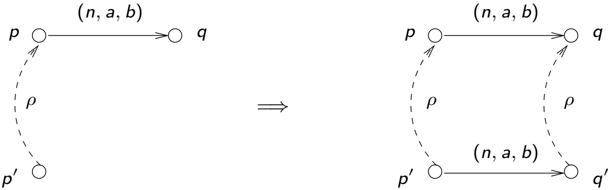
Illustrating condition (RC)

(RC-init):



and $b = b'$

(RC-sim):



Condition (RC) is necessary and sufficient for refinement

Theorem

Let $\mathcal{A}$ and $\mathcal{A}'$ be two ADT's of type N. Then $\mathcal{A}' \preceq \mathcal{A}$ iff they satisfy condition (RC).

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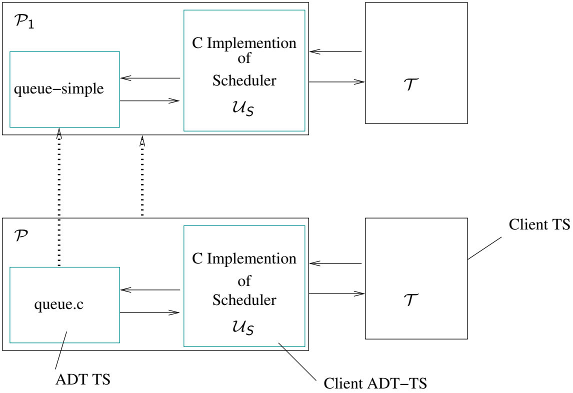
Exercise

Find an abstraction relation ρ for which $QADT_2$ and $QADT_3$ satisfy condition (RC).

Why ADT Transition Systems

- To reason about imperative implementations of ADT's (read transition-system based implementions)
- To do so compositionally.



A C implementation of a queue

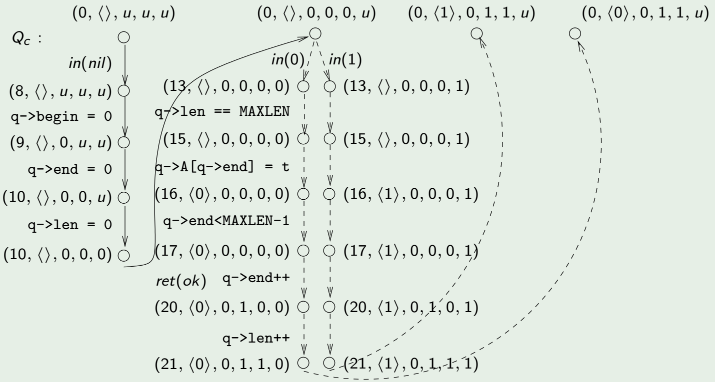
```
int A[MAXLEN];
unsigned beg, end, len;

void init() {
  beg = 0;
  end = 0;
  len = 0;
}

int deq() { ...  }
void enq(int t) {
  if (len == MAXLEN)
    assert(0); /* exception */
  A[end] = t;
  if (end < MAXLEN-1)
    end++;
  else
    end = 0;
  len++;
}
```

Example: ADT Transition System induced by queue.c

Part of the ADT TS induced by queue.c, showing init and enq opns



ADT induced by an ADT TS

An ADT transition system like $\mathcal{S}$ above induces an ADT $\mathcal{A}_{\mathcal{S}}$ of type N given by $\mathcal{A}_{\mathcal{S}} = (Q_c \cup \{E\}, U, E, \{op_n\}_{n \in N})$ where for each $n \in N$, $p \in Q_c \cup \{E\}$, and $a \in I_n$, we have:

$$op_n(p, a) = \begin{cases} (q, b) & \text{if there exists a path of the form} \\ & p \xrightarrow{in(a)} r_1 \xrightarrow{l_1} \cdots \xrightarrow{l_{k-1}} r_k \xrightarrow{ret(b)} q \text{ in } \mathcal{S} \\ (E, e) & \text{otherwise.} \end{cases}$$

We say that an ADT TS $\mathcal{S}'$ refines another ADT TS $\mathcal{S}$ iff $\mathcal{A}_{\mathcal{S}'}$ refines $\mathcal{A}_{\mathcal{S}}$.

Substitutability Claim

We claim that refinement is "substitutive" and gives us a compositional way of reasoning about ADT implementations:

Theorem

Let $\mathcal{U}$ be an M-client ADT transition system of type N, and $\mathcal{B}$ and $\mathcal{C}$ be ADT's of type M such that $\mathcal{C} \preceq \mathcal{B}$. Then we have $\mathcal{A}_{\mathcal{U}[\mathcal{C}]} \preceq \mathcal{A}_{\mathcal{U}[\mathcal{B}]}$.

Refinement Conditions in VCC: Using a ghost model

(init-a) $\mathtt{func}_{init}^{\mathcal{M},\mathcal{P}}$ terminates on all joint state-input pairs satisfying $pre_{init}^{\mathcal{M}}$.

(init-b) $\mathtt{func}_{init}^{\mathcal{M},\mathcal{P}}$ $(\mathtt{X}_{init}\ \mathtt{x})$

$\mathtt{\_(requires}\ pre_{init}^{\mathcal{M}}\mathtt{)}$

$\mathtt{\_(ensures}\ inv_{\rho} \wedge y_{init}^{\mathcal{M}} = y_{init}^{\mathcal{P}}\mathtt{)}$ {

  // body of $\mathtt{func}_{init}^{\mathcal{M}}$

  // body of $\mathtt{func}_{init}^{\mathcal{P}}$

}

(sim-a) For each operation n, $\mathtt{func}_{n}^{\mathcal{M},\mathcal{P}}$ must terminate on all state-input pairs satisfying $pre_{n}^{\mathcal{M}} \wedge inv_{\rho}$.

(sim-b) For each operation n:

$\mathtt{func}_{n}^{\mathcal{M},\mathcal{P}}$ $(\mathtt{X}_{n}\ \mathtt{x})$

$\mathtt{\_(requires}\ pre_{n}^{\mathcal{M}} \wedge inv_{\rho}\mathtt{)}$

$\mathtt{\_(ensures}\ inv_{\rho} \wedge y_{n}^{\mathcal{M}} = y_{n}^{\mathcal{P}}\mathtt{)}$ {

  // body of $\mathtt{func}_{n}^{\mathcal{M}}$

  // body of $\mathtt{func}_{n}^{\mathcal{P}}$

}

Some examples: store with get/put

```
struct store {
   int val1;
   int val2;
   int flag;
} s;

void init() {
   s.flag = 1;
   s.val1 = 0;
}

void set(int x) {
   if (s.flag)
       s.val2 = x;
   else
       s.val1 = x;
   s.flag = (1 - s.flag);
}
```

```
int get(void) {
    if (s.flag)
        return s.val1;
    else
        return s.val2;
 }
```

Overall strategy for refinement